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Quantum Optics
Winter semester 2018/2019 - Exercise sheet 3
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Problem 1: Coherent states of light.

a) Compute the photon number fluctuations $\langle(\hat{a}^\dagger\hat{a})^2 - \langle\hat{a}^\dagger\hat{a}\rangle^2\rangle$ for a coherent state $|\alpha\rangle$. Compare this value with the photon number expectation value for this same state.

b) Given the displacement operator $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, for which $|\alpha\rangle = D(\alpha)|0\rangle$, use the Baker-Campbell-Hausdorff formula to show that:

$$\hat{a}^\dagger|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle \quad .$$

c) Show that:

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp(i\mathrm{Im}\{\alpha\beta^*\}) \quad .$$

Problem 2: Algebra of bosonic operators.

a) Given two operator-valued functions $\hat{F}(\hat{a},\hat{a}^\dagger)$ and $\hat{G}(z;\hat{a},\hat{a}^\dagger) = e^{\hat{a}z}\hat{F}(\hat{a},\hat{a}^\dagger)e^{-\hat{a}z} = \hat{F}(\hat{a}, e^{\hat{a}z}\hat{a}^\dagger e^{-\hat{a}z})$, where $z \in \mathbb{C}$ and $\hat{F}$ is an arbitrary power series in $\hat{a}$ and $\hat{a}^\dagger$, show that from the derivative of $\hat{G}$ relative to z for $\hat{F} = \hat{a}^\dagger$ one can conclude that $d\hat{G}(z;\hat{a},\hat{a}^\dagger)/dz = [\hat{a}, \hat{F}(\hat{a},\hat{a}^\dagger + z)]$ and $\hat{G}(z;\hat{a},\hat{a}^\dagger) = \hat{F}(\hat{a},\hat{a}^\dagger + z)$ for any $\hat{F}$. Using these relations, show that $\partial\hat{F}(\hat{a},\hat{a}^\dagger)/\partial\hat{a}^\dagger = [\hat{a},\hat{F}(\hat{a},\hat{a}^\dagger)]$.

b) Consider the operator-valued functions $\hat{G}(\hat{a},\hat{a}^\dagger)$ and $\hat{F}(\hat{a},\hat{a}^\dagger) = \hat{a}\hat{G}(\hat{a},\hat{a}^\dagger)$. Knowing that such functions can be rearranged in any desired way with respect to the order of $\hat{a}$ and $\hat{a}^\dagger$ by means of the commutation relations, consider the function obtained from $\hat{G}$ by moving all anihilation operators to the right and the creation operators to the left, which is usually denoted by $\hat{G}^{(N)}$ ($\hat{G}^{(N)} = \hat{G}$) and called "normal ordered form" of the given function. Use the relation obtained in section (a) to show that $\hat{F}(\hat{a},\hat{a}^\dagger) = \partial\hat{G}^{(N)}(\hat{a},\hat{a}^\dagger)/\partial\hat{a}^\dagger + \hat{G}^{(N)}(\hat{a},\hat{a}^\dagger)\hat{a}$. Considering the "normal order operator" $\mathcal{N}\hat{A} =: \hat{A} :$, which when applied on a given operator function $\hat{A}(\hat{a},\hat{a}^\dagger)$ forcibly arranges all annihilation operators to the right and the creation operators to the left without accounting for the commutation relations (i.e., $:\hat{a}\hat{a}^\dagger := \hat{a}^\dagger\hat{a}$ and $\hat{G}^{(N)} \neq :\hat{G}:$), and replacing $\hat{a}$ for $\hat{a} + \partial/\partial\hat{a}^\dagger$ within the normal order operator, show that $\hat{F}^{(N)}(\hat{a},\hat{a}^\dagger) =: \hat{F}(\hat{a} + \partial/\partial\hat{a}^\dagger, \hat{a}^\dagger) :$.

Problem 3: Schrödinger-Robertson uncertainty relation.

Given an arbitrary state $|\Psi\rangle$ and two arbitrary Hermitian operators $\hat{A}$ and $\hat{B}$, show that, for $\sigma_A = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)^2|\Psi\rangle$ and $\sigma_B = \langle\Psi|(\hat{B} - \langle\hat{B}\rangle)^2|\Psi\rangle$, the following relation is true:

$$\sigma_A^2\sigma_B^2 \geq \frac{1}{4}\left|\langle\{\hat{A},\hat{B}\}\rangle - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2 + \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2.$$